

Linear-Quadratic Optimal Control Problems for Mean-Field Stochastic Differential Equations with Jumps *

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Abstract

In this paper, we study a linear- quadratic optimal control problem for mean-field stochastic differential equations driven by a Poisson random martingale measure and a multidimensional Brownian motion. Firstly, the existence and uniqueness of the optimal control is obtained by the classic convex variation principle. Secondly, by the duality method, the optimality system, also called the stochastic Hamilton system which turns out to be a linear fully coupled mean-field forward-backward stochastic differential equation with jumps, is derived to characterize the optimal control. Thirdly, applying a decoupling technique, we establish the connection between two Riccati equation and the stochastic Hamilton system and then prove the optimal control has a state feedback representation.

1 Introduction

1.1 Notations

Let T be a fixed strictly positive real number and $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{0 \leq t \leq T}, P)$ be a complete probability space on which a d -dimensional standard Brownian motion $\{W(t), 0 \leq t \leq T\}$ is defined. Denote by \mathcal{P} the \mathcal{F}_t -predictable σ -field on $[0, T] \times \Omega$ and by $\mathcal{B}(\Lambda)$ the Borel σ -algebra of any topological space Λ . Let $(Z, \mathcal{B}(Z), v)$ be a measurable space with $v(Z) < \infty$ and $\eta : \Omega \times D_\eta \rightarrow Z$ be an \mathcal{F}_t -adapted stationary Poisson point process with characteristic measure v , where D_η is a countable subset of $(0, \infty)$. Then the counting measure induced by η is

$$\mu((0, t] \times A) := \#\{s \in D_\eta; s \leq t, \eta(s) \in A\}, \quad \text{for } t > 0, A \in \mathcal{B}(Z).$$

And $\tilde{\mu}(de, dt) := \mu(de, dt) - v(de)dt$ is a compensated Poisson random martingale measure which is assumed to be independent of Brownian motion $\{W(t), 0 \leq t \leq T\}$. Assume $\{\mathcal{F}_t\}_{0 \leq t \leq T}$ is P -completed natural filtration generated by $\{W(t), 0 \leq t \leq T\}$ and $\{\iint_{A \times (0, t]} \tilde{\mu}(de, ds), 0 \leq t \leq T, A \in \mathcal{B}(Z)\}$. In the following, we introduce the basic notation used throughout this paper.

- H : a Hilbert space with norm $\|\cdot\|_H$.
- $\langle \alpha, \beta \rangle$: the inner product in $\mathbb{R}^n, \forall \alpha, \beta \in \mathbb{R}^n$.
- $|\alpha| = \sqrt{\langle \alpha, \alpha \rangle}$: the norm of $\mathbb{R}^n, \forall \alpha \in \mathbb{R}^n$.
- $\langle A, B \rangle = \text{tr}(AB^\top)$: the inner product in $\mathbb{R}^{n \times m}, \forall A, B \in \mathbb{R}^{n \times m}$. Here denote by B^\top the transpose of a matrix B .

- $|A| = \sqrt{\text{tr}(AA^\top)}$: the norm of A .
- S^n : the set of all $n \times n$ symmetric matrices.
- S_+^n : the subset of all non-negative definite matrices of S^n .

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- $S_{\mathcal{F}}^2(0, T; H)$: the space of all H -valued and \mathcal{F}_t -adapted càdlàg processes $f = \{f(t, \omega), (t, \omega) \in [0, T] \times \Omega\}$ satisfying

$$\|f\|_{S_{\mathcal{F}}^2(0, T; H)}^2 \triangleq \mathbb{E} \sup_{0 \leq t \leq T} \|f(t)\|_H^2 < +\infty.$$

- $M_{\mathcal{F}}^2(0, T; H)$: the space of all H -valued and \mathcal{F}_t -adapted processes $f = \{f(t, \omega), (t, \omega) \in [0, T] \times \Omega\}$ satisfying

$$\|f\|_{M_{\mathcal{F}}^2(0, T; H)}^2 \triangleq \mathbb{E} \left[\int_0^T \|f(t)\|_H^2 dt \right] < \infty.$$

- $M^{\nu, 2}(Z; H)$: the space of all H -valued measurable functions $r = \{r(\theta), \theta \in Z\}$ defined on the measure space $(Z, \mathcal{B}(Z); \nu)$ satisfying

$$\|r\|_{M^{\nu, 2}(Z; H)}^2 \triangleq \int_Z \|r(\theta)\|_H^2 \nu(d\theta) < \infty.$$

- $M_{\mathcal{F}}^{\nu, 2}([0, T] \times Z; H)$: the space of all $M^{\nu, 2}(Z; H)$ -valued and \mathcal{F}_t -predictable processes $r = \{r(t, \omega, e), (t, \omega, e) \in [0, T] \times \Omega \times Z\}$ satisfying

$$\|r\|_{M_{\mathcal{F}}^{\nu, 2}([0, T] \times Z; H)}^2 \triangleq \mathbb{E} \left[\int_0^T \|r(t, \cdot)\|_{M^{\nu, 2}(Z; H)}^2 dt \right] < \infty.$$

- $L^2(\Omega, \mathcal{F}, P; H)$: the space of all H -valued random variables ξ on (Ω, \mathcal{F}, P) satisfying

$$\|\xi\|_{L^2(\Omega, \mathcal{F}, P; H)}^2 \triangleq \mathbb{E}[\|\xi\|_H^2] < \infty.$$

1.2 Formulation of Problem

Consider the following linear stochastic system driven by Brownian motion $\{W(t)\}_{0 \leq t \leq T}$ and Poisson random martingale measure $\{\tilde{\mu}(d\theta, dt)\}_{0 \leq t \leq T}$

$$\begin{cases} dX(t) = (A(t)X(t) + \bar{A}(t)\mathbb{E}[X(t)] + B(t)u(t) + \bar{B}(t)\mathbb{E}[u(t)])dt \\ \quad + (C(t)X(t) + \bar{C}(t)\mathbb{E}[X(t)] + D(t)u(t) + \bar{D}(t)\mathbb{E}[u(t)])dW(t) \\ \quad + \int_Z (E(t, \theta)X(t-) + \bar{E}(t, \theta)\mathbb{E}[X(t-)] + F(t, \theta)u(t) + \bar{F}(t, \theta)\mathbb{E}[u(t)])\tilde{\mu}(d\theta, dt), \\ x(0) = x \in \mathbb{R}^n, \end{cases} \quad (1.1)$$

with the following quadratic cost functional

$$\begin{aligned} J(x, u(\cdot)) = & \mathbb{E} \left[\int_0^T \left(\langle Q(t)X(t), X(t) \rangle + \langle \bar{Q}(t)\mathbb{E}[X(t)], \mathbb{E}[X(t)] \rangle + \langle N(t)u(t), u(t) \rangle \right. \right. \\ & \left. \left. + \langle \bar{N}(t)\mathbb{E}[u(t)], \mathbb{E}[u(t)] \rangle \right) dt \right] + \mathbb{E}[\langle GX(T), X(T) \rangle] + \langle \bar{G}\mathbb{E}[X(T)], \mathbb{E}[X(T)] \rangle, \end{aligned} \quad (1.2)$$

where $A(\cdot), \bar{A}(\cdot), B(\cdot), \bar{B}(\cdot), C(\cdot), \bar{C}(\cdot), D(\cdot), \bar{D}(\cdot), E(\cdot, \cdot), \bar{E}(\cdot, \cdot), F(\cdot, \cdot), \bar{F}(\cdot), Q(\cdot), \bar{Q}(\cdot), N(\cdot), \bar{N}(\cdot)$ are given matrix valued deterministic functions, and G and \bar{G} are given matrices.

In the above, $u(\cdot)$ is our admissible control process. In this paper, a predictable stochastic process $u(\cdot)$ is said to be an admissible control, if $u(\cdot) \in M_{\mathcal{F}}^2(0, T; \mathbb{R}^m)$. The set of all admissible controls is denoted by \mathcal{A} . For any admissible control $u(\cdot)$, the strong solution of the system (1.1), denoted by $X^{(x, u)}(\cdot)$ or $X(\cdot)$ if its dependence on admissible control $u(\cdot)$ is clear from the context, is called the state process corresponding to the control process $u(\cdot)$, and $(u(\cdot), X(\cdot))$ is called an admissible pair.

Our optimal control problem can be stated as follows:

Problem 1.1. For given $x \in \mathbb{R}^n$, find an admissible control $u^*(\cdot)$ such that

$$J(x, u^*(\cdot)) = \inf_{u(\cdot) \in \mathcal{A}} J(x, u(\cdot)). \quad (1.3)$$

Any $u^*(\cdot) \in \mathcal{A}$ satisfying the above is called an optimal control process of Problem 1.1 and the corresponding state process $X^*(\cdot)$ is called the optimal state process. Correspondingly $(u^*(\cdot), X^*(\cdot))$ is called an optimal pair of Problem 1.1.

Note that $\mathbb{E}[X(\cdot)]$ and $\mathbb{E}[u(\cdot)]$ appear in the state equation and cost functional. Such a state equation is referred to as a mean-field stochastic differential equation (MF-SDE). For details on motivations for the inclusion of $\mathbb{E}[X(\cdot)]$ and $\mathbb{E}[u(\cdot)]$ in the cost functional, the interested reader is referred to [28].

Throughout this paper, we make the following assumptions on the coefficients

Assumption 1.1. The matrix-valued functions $A, \bar{A}, C, \bar{C}, Q, \bar{Q} : [0, T] \rightarrow \mathbb{R}^{n \times n}$; $B, \bar{B}, D, \bar{D} : [0, T] \rightarrow \mathbb{R}^{n \times m}$; $E, \bar{E} : [0, T] \rightarrow \mathcal{L}^{v,2}(Z; \mathbb{R}^{n \times n})$, $F, \bar{F} : [0, T] \rightarrow \mathcal{L}^{v,2}(Z; \mathbb{R}^{n \times m})$; $N, \bar{N} : [0, T] \rightarrow \mathbb{R}^{m \times m}$ are uniformly bounded measurable functions.

Assumption 1.2. The matrix-valued functions $Q, Q + \bar{Q}, N, N + \bar{N}$ are a.e. nonnegative matrix, and $M, M + \bar{M}$ are nonnegative matrices. Moreover, $N, N + \bar{N}$ uniformly positive, i.e. for $\forall u \in \mathbb{R}^m$ and a.s. $t \in [0, T]$, $\langle N(t)u, u \rangle \geq \delta \langle u, u \rangle$ and $\langle (N(t) + \bar{N}(t))u, u \rangle \geq \delta \langle u, u \rangle$, for some positive constant δ .

The following result gives the well-posedness of the state equation as well as some useful estimates.

Lemma 1.2. *Let Assumption 1.1 be satisfied. Then for any admissible control $u(\cdot)$, the state equation (1.1) has a unique solution $X(\cdot) \in S_{\mathcal{F}}^2(0, T; \mathbb{R}^n)$. Moreover, we have the following estimate*

$$\mathbb{E} \left[\sup_{0 \leq t \leq T} |X(t)|^2 \right] \leq K \mathbb{E} \left[\int_0^T |u(t)|^2 dt + x \right] \quad (1.4)$$

and

$$|J(x, u(\cdot))| < \infty. \quad (1.5)$$

Suppose that $\bar{X}(\cdot)$ be the state process corresponding to another admissible control $\bar{u}(\cdot)$, then we have the following estimate

$$\mathbb{E} \left[\sup_{0 \leq t \leq T} |X(t) - \bar{X}(t)|^2 \right] \leq K \mathbb{E} \left[\int_0^T |u(t) - \bar{u}(t)|^2 dt \right]. \quad (1.6)$$

Proof. The existence and uniqueness of the solution can be obtained by a standard argument using the contraction mapping theorem. For the estimates (1.4) and (1.6), we can easily obtain them by applying the Itô formula to $|X(\cdot)|^2$ and $|X(\cdot) - \bar{X}(\cdot)|^2$, Gronwall inequality and B-D-G inequality. For the estimate (1.5), using Assumption 1.1 and the estimate (1.4), we have

$$|J(x, u(\cdot))| \leq K \left\{ \mathbb{E} \left[\sup_{0 \leq t \leq T} |X(t)|^2 \right] + \mathbb{E} \left[\int_0^T |u(t)|^2 dt \right] \right\} \leq K \mathbb{E} \left[\int_0^T |u(t)|^2 dt + x \right] < \infty. \quad (1.7)$$

where we have used the elementary inequality: for any $\xi \in L^2(\Omega, \mathcal{F}, P; H)$

$$\|\mathbb{E}\xi\|_H^2 \leq \mathbb{E}\|\xi\|_H^2 \quad (1.8)$$

The proof is complete. \square

Therefore, by Lemma 1.2, we know that Problem 1.1 is well-defined.

1.3 Related Development and Contributions of this paper

Most recently, thanks to comprehensive practical applications such as in economics and finance, optimal control problems of mean-field type become a popular topic and are studied by many researcher. The main feature of this type of problems is that the coefficients of the state equation and cost functional depend not only on the state and the control but also on their probability distribution. As described in [4], due to the mean-field term involved in the cost functional, the corresponding optimal control problem become to be a time-inconsistent optimal control problem where the dynamic programming principle (DPP) is not effective which makes many researchers to solve this type of optimal control problems by establishing the stochastic maximum principle (SMP) instead of trying extensions of DPP. Interested readers may refer to [1-6],[9],[10], [12],[16], [17], [22-25],[27] for various versions of the stochastic maximum principles for the mean-field models.

In 2013, Yong [28] systematically studied the continuous-time mean-field LQ control problem, where the optimal control is represented as a state feedback form by introducing two Riccati differential equations. Since [28], abundant results and advances have been made on mean-field LQ control problem (cf., for example, [8], [11], [18-21],[26]). Different the above mentioned references, the purpose of this paper is to extend continuous-time mean-field LQ control problem to jump diffusion system (1.1) and establish the corresponding theoretical results. We first establish the existence and uniqueness of the optimal control by classic convex variation principle in Section 2. Then, in Section 3, we will establish the dual characterization of the optimal control by optimality system, also called stochastic Hamiltonian system. Here the stochastic Hamiltonian system turns out to be a coupled forward-backward stochastic differential equation of mean-field type with jump, consisting of the state equation, the adjoint equation and the dual presentation of the optimal control. Although the stochastic Hamilton system gives a complete characterization of the stochastic LQ problem, it is a fully coupled forward-backward stochastic differential equation, which is very difficult, if not possible, to be solved. Meanwhile, it is natural to link the stochastic LQ problem with Riccati equation. In section 4, we will introduce two Riccati equation and establish its connection with the stochastic Hamilton system, and then prove the optimal control has state feedback representation.

2 Existence and Uniqueness of Optimal Control

In this section, we study the existence and uniqueness of the optimal control of Problem 1.1. To this end, We first establish some elementary properties of the cost functional.

Lemma 2.1. *Let Assumptions 1.1 and 1.2 be satisfied. Then the cost functional $J(x, u(\cdot))$ is continuous over \mathcal{A} .*

Proof. Let $(u(\cdot), X(\cdot))$ and $(\bar{u}(\cdot), \bar{X}(\cdot))$ be any two admissible control pairs. Under Assumptions 1.1 and 1.2, from the definition of the cost functional $J(x, u(\cdot))$ (see (1.2)), we have

$$\begin{aligned} |J(u(\cdot)) - J(\bar{u}(\cdot))|^2 &\leq K \left\{ \mathbb{E} \left[\int_0^T |u(t) - \bar{u}(t)|^2 dt \right] + \mathbb{E} \left[\int_0^T |X(t) - \bar{X}(t)|^2 dt \right] \right\} \\ &\quad \times \left\{ \mathbb{E} \left[\int_0^T |u(t)|^2 dt \right] + \mathbb{E} \left[\int_0^T |X(t)|^2 dt \right] + \mathbb{E} \left[\int_0^T |\bar{u}(t)|^2 dt \right] + \mathbb{E} \left[\int_0^T |\bar{X}(t)|^2 dt \right] \right\}. \end{aligned} \quad (2.1)$$

Using the estimates (1.4) and (1.6) lead to

$$|J(u(\cdot)) - J(\bar{u}(\cdot))|^2 \leq K \left\{ \mathbb{E} \left[\int_0^T |u(t) - \bar{u}(t)|^2 dt \right] \right\} \times \left\{ \mathbb{E} \left[\int_0^T |u(t)|^2 dt \right] + \mathbb{E} \left[\int_0^T |\bar{u}(t)|^2 dt \right] + x \right\}. \quad (2.2)$$

Thus, it follows that

$$J(u(\cdot)) - J(\bar{u}(\cdot)) \rightarrow 0, \quad \text{as } u(\cdot) \rightarrow \bar{u}(\cdot) \text{ in } \mathcal{A}. \quad (2.3)$$

The proof is complete. \square

Lemma 2.2. *Let Assumptions 1.1 and 1.2 be satisfied. Then the cost functional $J(x, u(\cdot))$ is strictly convex \mathcal{A} . Moreover, the cost functional $J(x, u(\cdot))$ is coercive over \mathcal{A} , i.e.,*

$$\lim_{\|u(\cdot)\|_{\mathcal{A}} \rightarrow \infty} J(x, u(\cdot)) = \infty.$$

Proof. Since the weighting matrices in the cost functional is not random, it is easy to check that

$$\begin{aligned} J(x, u(\cdot)) = & \mathbb{E} \left[\int_0^T \left(\langle Q(t)(X(t) - \mathbb{E}[X(t)]), X(t) - \mathbb{E}[X(t)] \rangle + \langle (Q + \bar{Q})(t) \mathbb{E}[X(t)], \mathbb{E}[X(t)] \rangle \right. \right. \\ & + \langle N(t)(u(t) - \mathbb{E}[u(t)]), u(t) - \mathbb{E}[u(t)] \rangle + \langle (N(t) + \bar{N}(t)) \mathbb{E}[u(t)], \mathbb{E}[u(t)] \rangle \Big) dt \Big] \\ & + \mathbb{E} \left[\langle M(X(T) - \mathbb{E}[X(T)]), X(T) - \mathbb{E}[X(T)] \rangle + \langle (M + \bar{M}) \mathbb{E}[X(T)], \mathbb{E}[X(T)] \rangle \right], \end{aligned} \quad (2.4)$$

Thus the cost functional $J(x, u(\cdot))$ over \mathcal{A} is convex from the nonnegativity of the $N, N + \bar{N}, Q, Q + \bar{Q}, M, M + \bar{M}$. Actually, since $N, N + \bar{N}$ is uniformly positive, $J(u(\cdot))$ is strictly convex. On the other hand, by Assumption 1.2 and (2.4), we get

$$\begin{aligned} J(x, u(\cdot)) & \geq \mathbb{E} \left[\int_0^T \left(\langle N_1(t)(u(t) - \mathbb{E}[u(t)]), u(t) - \mathbb{E}[u(t)] \rangle + \langle (N_1(t) + N_2(t)) \mathbb{E}[u(t)], \mathbb{E}[u(t)] \rangle \right) dt \right] \\ & \geq \delta \mathbb{E} \left[\int_0^T \langle u(t) - \mathbb{E}[u(t)], u(t) - \mathbb{E}[u(t)] \rangle dt \right] + \delta \mathbb{E} \left[\int_0^T \langle \mathbb{E}[u(t)], \mathbb{E}[u(t)] \rangle dt \right] \\ & = \delta \mathbb{E} \left[\int_0^T |u(t)|^2 dt \right] \\ & = \delta \|u(\cdot)\|_{\mathcal{A}}^2. \end{aligned} \quad (2.5)$$

Thus $\lim_{\|u(\cdot)\|_{\mathcal{A}} \rightarrow \infty} J(x, u(\cdot)) = \infty$. The proof is complete. \square

Lemma 2.3. *Let Assumptions 1.1 and 1.2 be satisfied. Then the cost functional $J(x, u(\cdot))$ is Fréchet differentiable over \mathcal{A} and the corresponding Fréchet derivative $J'(x, u(\cdot))$ is given by*

$$\begin{aligned} \langle J'(x, u(\cdot)), v(\cdot) \rangle = & 2 \mathbb{E} \left[\int_0^T \left(\langle Q(t)X^{(x,u)}(t), X^{(0,v)}(t) \rangle + \langle \bar{Q}(t) \mathbb{E}[X^{(x,u)}(t)], \mathbb{E}[X^{(0,v)}(t)] \rangle \right. \right. \\ & + \langle N(t)u(t), v(t) \rangle + \langle \bar{N}(t) \mathbb{E}[u(t)], \mathbb{E}[v(t)] \rangle \Big) dt \Big] + 2 \mathbb{E} \left[\langle M X^{(x,u)}(T), X^{(0,v)}(T) \rangle \right. \\ & \left. + 2 \langle \bar{M} \mathbb{E}[X^{(x,u)}(T)], \mathbb{E}[X^{(0,v)}(T)] \rangle \right], \quad \forall u(\cdot), v(\cdot) \in \mathcal{A}, \end{aligned} \quad (2.6)$$

where $X^{(0,v)}(\cdot)$ is the solution of the state equation (1.1) corresponding to the admissible control $v(\cdot)$ and the initial value $X(0) = 0$, and $X^{(x,u)}(\cdot)$ is the state process corresponding to the control process $u(\cdot)$ with the initial value $X(0) = x$.

Proof. Let $u(\cdot)$ and $v(\cdot)$ be two any given admissible control. For simplicity, the right hand side of (2.6) is denoted by $\Delta^{u,v}$. Since the state equation (1.1) is linear, it is easily to check that

$$X^{x,u+v}(t) = X^{x,u}(t) + X^{0,v}(t), 0 \leq t \leq T. \quad (2.7)$$

For simplicity, the right hand side of (2.6) is denoted by $\Delta^{u,v}$. Therefore, in terms of (2.7) and the definition of the cost functional $J(x, u(\cdot))$ (see (1.2)), it is easy to check that

$$J(x, u(\cdot) + v(\cdot)) - J(x, u(\cdot)) = J(0, v(\cdot)) + \Delta^{u,v} \quad (2.8)$$

On the other hand, the estimate (1.7) leads to

$$|J(0, v(\cdot))| \leq K \mathbb{E} \left[\int_0^T |v(t)|^2 dt \right] = K \|v(\cdot)\|_{\mathcal{A}}^2, \quad (2.9)$$

Therefore,

$$\lim_{\|v(\cdot)\|_{\mathcal{A}} \rightarrow 0} \frac{|J(x, u(\cdot) + v(\cdot)) - J(x, u(\cdot)) - \Delta^{u,v}|}{\|v(\cdot)\|_{\mathcal{A}}} = \lim_{\|v(\cdot)\|_{\mathcal{A}} \rightarrow 0} \frac{|J(0, v(\cdot))|}{\|v(\cdot)\|_{\mathcal{A}}} = 0 \quad (2.10)$$

which gives that $J(x, u(\cdot))$ has Fréchet derivative $\Delta^{u,v}$. The proof is complete. \square

Remark 2.1. Since the cost function $J(x, u(\cdot))$ is Fréchet differentiable, then it is also Gâteaux differentiable. Moreover, the Gâteaux derivative is the Fréchet derivative $\langle J'(u(\cdot)), v(\cdot) \rangle$. In fact, from (2.8), we have

$$\begin{aligned} & \lim_{\varepsilon \rightarrow 0} \frac{J(x, u(\cdot) + \varepsilon v(\cdot)) - J(x, u(\cdot))}{\varepsilon} \\ &= \lim_{\varepsilon \rightarrow 0} \frac{J(0, \varepsilon v(\cdot)) + \Delta^{u, \varepsilon v}}{\varepsilon} \\ &= \lim_{\varepsilon \rightarrow 0} \frac{\varepsilon^2 J(0, v(\cdot)) + \varepsilon \Delta^{u,v}}{\varepsilon} \\ &= \Delta^{u,v} \\ &= \langle J'(u(\cdot)), v(\cdot) \rangle \end{aligned} \quad (2.11)$$

Now by Lemma 2.1-2.3, we can obtain the existence and uniqueness of optimal control. This result is stated as follows.

Theorem 2.4. *Let Assumptions 1.1 and 1.2 be satisfied. Then Problem 1.1 has a unique optimal control.*

Proof. Since the admissible controls set $\mathcal{A} = M_{\mathcal{F}}^2(0, T; \mathbb{R}^m)$ is a reflexive Banach space, in terms of Lemma 2.1-2.3, the uniqueness and existence of the optimal control of Problem 1.1 can be directly got from Proposition 2.12 of [7] (i.e., the coercive, strictly convex and lower-semi continuous functional defined on the reflexive Banach space has a unique minimum value point.) The proof is complete. \square

Theorem 2.5. *Let Assumptions 1.1 and 1.2 be satisfied. Then a necessary and sufficient condition for an admissible control $u(\cdot) \in \mathcal{A}$ to be an optimal control of Problem 1.1 is that for any admissible control $v(\cdot) \in \mathcal{A}$,*

$$\langle J'(x, u(\cdot)), v(\cdot) \rangle = 0, \quad (2.12)$$

i.e.

$$\begin{aligned} 0 = & 2\mathbb{E} \left[\int_0^T \left(\langle Q(t)X^{(x,u)}(t), X^{(0,v)}(t) \rangle + \langle \bar{Q}(t)\mathbb{E}[X^{(x,u)}(t)], \mathbb{E}[X^{(0,v)}(t)] \rangle \right. \right. \\ & \left. \left. + \langle N(t)u(t), v(t) \rangle + \langle \bar{N}(t)\mathbb{E}[u(t)], \mathbb{E}[v(t)] \rangle \right) dt \right] + 2\mathbb{E} \left[\langle MX^{(x,u)}(T), X^{(0,v)}(T) \rangle \right. \\ & \left. + 2\langle \bar{M}\mathbb{E}[X^{(x,u)}(T)], \mathbb{E}[X^{(0,v)}(T)] \rangle \right], \quad \forall u(\cdot), v(\cdot) \in \mathcal{A}. \end{aligned} \quad (2.13)$$

Proof. For the necessary part, suppose that $u(\cdot)$ is an optimal control. Then from (2.11), for any admissible control $v(\cdot)$, we have

$$\langle J'(x, u(\cdot)), v(\cdot) \rangle = \lim_{\varepsilon \rightarrow 0^+} \frac{J(x, u(\cdot) + \varepsilon v(\cdot)) - J(x, u(\cdot))}{\varepsilon} \geq 0, \quad (2.14)$$

and

$$-\langle J'(x, u(\cdot)), v(\cdot) \rangle = \langle J'(x, u(\cdot)), -v(\cdot) \rangle = \lim_{\varepsilon \rightarrow 0^+} \frac{J(x, u(\cdot) + \varepsilon(-v(\cdot))) - J(x, u(\cdot))}{\varepsilon} \geq 0, \quad (2.15)$$

which imply that

$$\langle J'(u(\cdot)), v(\cdot) \rangle = 0. \quad (2.16)$$

For the sufficient part, let $u(\cdot)$ be an given admissible control, and suppose that for any admissible control $v(\cdot)$, $\langle J'(u(\cdot)), v(\cdot) \rangle = 0$. Since the cost functional J is convex, then we have

$$J(x, v(\cdot)) - J(x, u(\cdot)) \geq \langle J'(x, u(\cdot)), v(\cdot) \rangle = 0. \quad (2.17)$$

which implies that $u(\cdot)$ is an optimal control. The proof is complete. \square

3 Optimality Conditions and Stochastic Hamilton Systems

Now we derive a necessary and sufficient condition for an admissible pair of Problem 1.1 to be an optimal pair by adjoint equation.

Theorem 3.1. *Let Assumptions 1.1 and 1.2 be satisfied. Then, a necessary and sufficient condition for an admissible pair $(u(\cdot); X(\cdot))$ to be an optimal pair of Problem is that the admissible control $u(\cdot)$ satisfies*

$$\begin{aligned} & 2N(t)u(t) + 2\bar{N}(t)\mathbb{E}[u(t)] + B^\top(t)p(t-) + \bar{B}^\top(t)\mathbb{E}[p(t-)] + D^\top(t)q(t) + \bar{D}^\top(t)\mathbb{E}[q(t)] \\ & + \int_Z F^\top(t, \theta)r(t, \theta)\nu(d\theta) + \int_Z \bar{F}^\top(t, \theta)\mathbb{E}[r(t, \theta)]\nu(d\theta) = 0, \quad a.e.a.s., \end{aligned} \quad (3.1)$$

where $(p(\cdot), q(\cdot), r(\cdot, \cdot))$ is the unique solution of the following mean-field backward SDE (MF-BSDE)

$$\begin{cases} dp(t) = - \left[A^\top(t)p(t) + \bar{A}(t)^\top \mathbb{E}[p(t)] + C^\top(t)q(t) + \bar{C}^\top(t)\mathbb{E}[q(t)] + \int_Z E^\top(t, \theta)r(t, \theta)\nu(d\theta) \right. \\ \quad \left. + \int_Z \bar{E}^\top(t, \theta)\mathbb{E}[r(t, \theta)]\nu(d\theta) + 2Q(t)X(t) + 2\bar{Q}(t)\mathbb{E}[X(t)] \right] dt + q(t)dW(t) + \int_Z r(t, \theta)\tilde{\mu}(d\theta, dt), \\ p(T) = 2GX(T) + 2\bar{G}\mathbb{E}[X(T)]. \end{cases} \quad (3.2)$$

Proof. Let $u(\cdot) \in \mathcal{A}$ be a given admissible control. Then for any admissible control $v(\cdot) \in \mathcal{A}$, from Lemma 2.3, we have

$$\begin{aligned} \langle J'(u(\cdot)), v(\cdot) \rangle &= 2\mathbb{E} \left[\int_0^T \left(\langle Q(t)X^{(x,u)}(t), X^{(0,v)}(t) \rangle + \langle \bar{Q}(t)\mathbb{E}[X^{(x,u)}(t)], \mathbb{E}[X^{(0,v)}(t)] \rangle \right. \right. \\ & \quad \left. \left. + \langle N(t)u(t), v(t) \rangle + \langle \bar{N}(t)\mathbb{E}[u(t)], \mathbb{E}[v(t)] \rangle \right) dt \right] + 2\mathbb{E}[\langle MX^{(x,u)}(T), X^{(0,v)}(T) \rangle] \\ & \quad + 2\langle \bar{M}\mathbb{E}[X^{(x,u)}(T)], \mathbb{E}[X^{(0,v)}(T)] \rangle. \end{aligned} \quad (3.3)$$

On the other hand, by [23], we know that (3.2) admits a unique adapted solution $(p(\cdot), q(\cdot), r(\cdot, \cdot))$. Applying Its

formula to $\langle X^{0,u}(t), q(t) \rangle$ and taking expectation, we have

$$\begin{aligned}
& 2\mathbb{E} \left[\int_0^T \left(\langle Q(t)X^{(x,u)}(t), X^{(0,v)}(t) \rangle + \langle \bar{Q}(t)\mathbb{E}[X^{(x,u)}(t)], \mathbb{E}[X^{(0,v)}(t)] \rangle \right) dt \right. \\
& \quad \left. + 2\mathbb{E} \left[\langle GX^{(x,u)}(T), X^{(0,v)}(T) \rangle + 2\langle \bar{G}\mathbb{E}[X^{(x,u)}(T)], \mathbb{E}[X^{(0,v)}(T)] \rangle \right] \right. \\
& = \mathbb{E} \left[\int_0^T \left(\langle p(t), B(t)v(t) + \bar{B}(t)\mathbb{E}[v(t)] \rangle + \langle q(t), D(t)v(t) + \bar{D}(t)\mathbb{E}[v(t)] \right. \right. \\
& \quad \left. \left. + \int_Z \langle r(t, \theta), F(t, \theta)v(t) + \bar{F}(t, \theta)\mathbb{E}[v(t)] \rangle \nu(d\theta) \right) dt \right] \\
& = \mathbb{E} \left[\int_0^T \left\langle B^\top(t)p(t-) + \bar{B}^\top(t)\mathbb{E}[p(t-)] + D^\top(t)q(t) + \bar{D}^\top(t)\mathbb{E}[q(t)] \right. \right. \\
& \quad \left. \left. + \int_Z F^\top(t, \theta)r(t, \theta)\nu(d\theta) + \int_Z \bar{F}^\top(t, \theta)\mathbb{E}[r(t, \theta)]\nu(d\theta), v(t) \right\rangle dt \right]
\end{aligned} \tag{3.4}$$

Putting (3.4) into (3.3), we get

$$\begin{aligned}
\langle J'(u(\cdot)), v(\cdot) \rangle &= \mathbb{E} \left[\int_0^T \left\langle 2N(t)u(t) + 2\bar{N}(t)\mathbb{E}[u(t)] + B^\top(t)p(t-) + \bar{B}^\top(t)\mathbb{E}[p(t-)] + D^\top(t)q(t) \right. \right. \\
& \quad \left. \left. + \bar{D}^\top(t)\mathbb{E}[q(t)] + \int_Z F^\top(t, \theta)r(t, \theta)\nu(d\theta) + \int_Z \bar{F}^\top(t, \theta)\mathbb{E}[r(t, \theta)]\nu(d\theta), v(t) \right\rangle dt \right]
\end{aligned} \tag{3.5}$$

For the necessary, let $(u(\cdot); X(\cdot))$ be an optimal pair, then from Theorem 2.5, we have $\langle J'(u(\cdot)), v(\cdot) \rangle = 0$ which imply that

$$\begin{aligned}
& 2N(t)u(t) + 2\bar{N}(t)\mathbb{E}[u(t)] + B^\top(t)p(t-) + \bar{B}^\top(t)\mathbb{E}[p(t-)] + D^\top(t)q(t) + \bar{D}^\top(t)\mathbb{E}[q(t)] \\
& \quad + \int_Z F^\top(t, \theta)r(t, \theta)\nu(d\theta) + \int_Z \bar{F}^\top(t, \theta)\mathbb{E}[r(t, \theta)]\nu(d\theta) = 0, a.e.a.s.,
\end{aligned} \tag{3.6}$$

from (3.5), since $v(\cdot)$ is arbitrary.

For the sufficient part, let $(u(\cdot); X(\cdot))$ be an admissible pair satisfying (3.1). Putting (3.1) into (3.5), then we have $\langle J'(u(\cdot)), v(\cdot) \rangle = 0$ which implies that $(u(\cdot); X(\cdot))$ is an optimal control pair from Theorem 2.5. \square

Finally we introduce the so-called stochastic Hamilton system which consists of the state equation (1.1), the adjoint equation (3.2) and the dual representation (3.1):

$$\left\{ \begin{aligned}
& dX(t) = [A(t)X(t) + \bar{A}(t)\mathbb{E}[X(t)] + B(t)u(t) + \bar{B}(t)\mathbb{E}[u(t)]]dt \\
& \quad + [C(t)X(t) + \bar{C}(t)\mathbb{E}[X(t)] + D(t)u(t) + \bar{D}(t)\mathbb{E}[u(t)]]dW(t) \\
& \quad + \int_Z [E(t, \theta)X(t-) + \bar{E}(t, \theta)\mathbb{E}[X(t-)] + F(t, \theta)u(t) + \bar{F}(t, \theta)\mathbb{E}[u(t)]]\tilde{\mu}(d\theta, dt), \\
& dp(t) = -[A^\top(t)p(t) + \bar{A}^\top(t)\mathbb{E}[p(t)] + C^\top(t)q(t) + \bar{C}^\top(t)\mathbb{E}[q(t)] + \int_Z E^\top(t, \theta)r(t, \theta)\nu(d\theta) \\
& \quad + \int_Z \bar{E}^\top(t, \theta)\mathbb{E}[r(t, \theta)]\nu(d\theta) + 2Q(t)X(t) + 2\bar{Q}(t)\mathbb{E}[X(t)]]dt + q(t)dW(t) + \int_Z r(t, \theta)\tilde{\mu}(d\theta, dt), \\
& X(0) = x, \quad p(T) = 2GX(T) + 2\bar{G}\mathbb{E}[X(T)], \\
& 2N(t)u(t) + 2\bar{N}(t)\mathbb{E}[u(t)] + B^\top(t)p(t-) + \bar{B}^\top(t)\mathbb{E}[p(t-)] + D^\top(t)q(t) + \bar{D}^\top(t)\mathbb{E}[q(t)] + \int_Z F^\top(t, \theta)r(t, \theta)\nu(d\theta) \\
& \quad + \int_Z \bar{F}^\top(t, \theta)\mathbb{E}[r(t, \theta)]\nu(d\theta) = 0.
\end{aligned} \right. \tag{3.7}$$

This is a fully coupled mean-field forward- backward stochastic differential equation (MF-FBSDE in short) and its solution consists of $(u(\cdot), X(\cdot), p(\cdot), q(\cdot), r(\cdot, \cdot))$.

Theorem 3.2. *Let Assumptions 1.1 and 1.2 be satisfied. Then stochastic Hamilton system (3.7) has a unique solution $(u(\cdot), X(\cdot), p(\cdot), q(\cdot), r(\cdot)) \in M_{\mathcal{F}}^2(0, T; \mathbb{R}^m) \times S_{\mathcal{F}}^2(0, T; \mathbb{R}^n) \times S_{\mathcal{F}}^2(0, T; \mathbb{R}^n) \times M_{\mathcal{F}}^{\nu, 2}([0, T] \times Z; \mathbb{R}^n)$. And $u(\cdot)$ is the unique optimal control of Problem 1.1 and $X(\cdot)$ is its corresponding optimal state process.*

Proof. By Theorem 2.4, Problem 1.1 admits a unique optimal pair $(u(\cdot), X(\cdot))$. Suppose $(p(\cdot), q(\cdot), r(\cdot, \cdot))$ is the unique solution of the adjoint equation (3.2) corresponding to the optimal pair $(u(\cdot); X(\cdot))$. Then by the necessary part of Theorem 3.1, the optimal control has the dual presentation (3.1). Consequently, $(u(\cdot), X(\cdot), p(\cdot), q(\cdot), r(\cdot, \cdot))$ consists of an adapted solution to the stochastic Hamilton system (3.7). Next, if the stochastic Hamilton system (3.7) has another adapted solution $(\bar{u}(\cdot), \bar{X}(\cdot), \bar{p}(\cdot), \bar{q}(\cdot), \bar{r}(\cdot, \cdot))$, then $(\bar{u}(\cdot), \bar{X}(\cdot))$ must be an optimal pair of Problem 1.1 by the sufficient part of Theorem 3.1. So we must have $u(\cdot) = \bar{u}(\cdot)$ by the uniqueness of the optimal control. Furthermore, by the uniqueness of the solutions of MF-SDE and MF-BSDE, one must have $(\bar{X}(\cdot), \bar{p}(\cdot), \bar{q}(\cdot), \bar{r}(\cdot, \cdot)) = (X(\cdot), p(\cdot), q(\cdot), r(\cdot, \cdot))$. The proof is complete. \square

Remark 3.1. In summary, the stochastic Hamilton system (3.7) completely characterizes the optimal control of Problem 1.1. Therefore, solving Problem 1.1 is equivalent to solving the stochastic Hamilton system, moreover, the unique optimal control can be given by (3.1). Taking expectation to (3.1), we have

$$\begin{aligned} & 2(N(t) + \bar{N}(t))\mathbb{E}[u(t)] + (B^\top(t) + \bar{B}^\top(t))\mathbb{E}[p(t-)] + (D^\top(t) + \bar{D}^\top(t))\mathbb{E}[q(t)] \\ & + \int_Z (F^\top(t, \theta) + \bar{F}^\top(t, \theta))\mathbb{E}[r(t, \theta)]\nu(d\theta) = 0, \quad a.e.a.s., \end{aligned} \quad (3.8)$$

which implies that

$$\begin{aligned} \mathbb{E}[u(t)] = & -\frac{1}{2}(N(t) + \bar{N}(t))^{-1} \left[(B(t) + \bar{B}(t))^\top \mathbb{E}[p(t-)] + (D(t) + \bar{D}(t))^\top \mathbb{E}[q(t)] \right. \\ & \left. + \int_Z (F(t, \theta) + \bar{F}(t, \theta))^\top \mathbb{E}[r(t, \theta)]\nu(d\theta) \right], \quad a.e.a.s. \end{aligned} \quad (3.9)$$

From (3.1), we know that

$$\begin{aligned} 2N(t)u(t) = & -2\bar{N}(t)\mathbb{E}[u(t)] - B^\top(t)p(t-) - \bar{B}^\top(t)\mathbb{E}[p(t-)] - D^\top(t)q(t) - \bar{D}^\top(t)\mathbb{E}[q(t)] \\ & - \int_Z F^\top(t, \theta)r(t, \theta)\nu(d\theta) - \int_Z \bar{F}^\top(t, \theta)\mathbb{E}[r(t, \theta)]\nu(d\theta), \quad a.e.a.s., \end{aligned} \quad (3.10)$$

Then putting (3.9) into (3.10), we have

$$\begin{aligned} u(t) = & -\frac{1}{2}N^{-1}(t) \left\{ B^\top(t)p(t-) + \bar{B}^\top(t)\mathbb{E}[p(t-)] + D^\top(t)q(t) + \bar{D}^\top(t)\mathbb{E}[q(t)] \right. \\ & \left. + \int_Z F^\top(t, \theta)r(t, \theta)\nu(d\theta) + \int_Z \bar{F}^\top(t, \theta)\mathbb{E}[r(t, \theta)]\nu(d\theta) \right\} \\ & + \bar{N}(t)(N(t) + \bar{N}(t))^{-1} \left[(B(t) + \bar{B}(t))^\top \mathbb{E}[p(t-)] + (D(t) + \bar{D}(t))^\top \mathbb{E}[q(t)] \right. \\ & \left. + \int_Z (F(t, \theta) + \bar{F}(t, \theta))^\top \mathbb{E}[r(t, \theta)]\nu(d\theta) \right] \Big\}, \quad a.e.a.s. \end{aligned} \quad (3.11)$$

4 Backward Riccati equation and State Feedback Representation of Optimal Control

Although the optimal control of Problem 1.1 is completely characterized by the stochastic Hamilton system (3.7), (3.7) is a fully coupled mean-field forward-backward stochastic differential equation whose solvability is much difficult to be obtained. Meanwhile, it is natural to link the stochastic LQ problem with Riccati equation. In this section, we will introduce two Riccati equation and establish its connection with the stochastic Hamilton system (3.7), and then prove the optimal control has state feedback representation.

4.1 Derivation of Riccati equations

Let $(u(\cdot), X(\cdot))$ be the optimal pair of Problem 1.1 associated with the adjoint process $(p(\cdot), q(\cdot), r(\cdot, \cdot))$ being the solution of the adjoint equation (3.2). This means $(u(\cdot), X(\cdot), p(\cdot), q(\cdot), r(\cdot, \cdot))$ is the solution of the stochastic Hamilton system (3.7).

Taking expectation on both sides of the stochastic Hamilton system (3.7), we get that $(\mathbb{E}[X(\cdot)], \mathbb{E}[p(\cdot)])$ satisfies the following forward-backward ordinary differential equation (suppressing s)

$$\begin{cases} d\mathbb{E}[X] &= \left[(A + \bar{A})\mathbb{E}[X] + (B + \bar{B})\mathbb{E}[u] \right] dt, \\ d\mathbb{E}[p] &= - \left[(A^\top + \bar{A}^\top)\mathbb{E}[p] + (C^\top + \bar{C}^\top)\mathbb{E}[q] + \int_Z (E^\top(\theta) + \bar{E}^\top(\theta))\mathbb{E}[r(t, \theta)]\nu(d\theta) + 2(Q + \bar{Q})\mathbb{E}[X] \right] dt \\ \mathbb{E}[p(T)] &= 2(G + \bar{G})\mathbb{E}[X(T)], \\ \mathbb{E}[X(0)] &= x. \end{cases} \quad (4.1)$$

Further, it is easy to check that $(X(\cdot) - \mathbb{E}[X(\cdot)], p(\cdot) - \mathbb{E}[p(\cdot)])$ satisfies the following forward-backward stochastic differential equation

$$\begin{cases} d(X - \mathbb{E}[X]) &= \left[A(X - \mathbb{E}[X]) + B(u - \mathbb{E}[u]) \right] dt + \left[C(X - \mathbb{E}[X]) + (C + \bar{C})\mathbb{E}[X] + D(u - \mathbb{E}[u]) + (D + \bar{D})\mathbb{E}[u] \right] dW(t) \\ &\quad + \int_Z \left[E(\theta)(X - \mathbb{E}[X]) + (E(\theta) + \bar{E}(\theta))\mathbb{E}[X] + F(\theta)(u - \mathbb{E}[u]) + (F(\theta) + \bar{F}(\theta))\mathbb{E}[u] \right] \tilde{\mu}(d\theta, dt), \\ d(p - \mathbb{E}[p]) &= - \left[A^\top (p - \mathbb{E}[p]) + C^\top (q - \mathbb{E}[q]) + \int_Z E^\top(t, \theta)(r(t, \theta) - \mathbb{E}[r(t, \theta)])\nu(d\theta) + 2Q(X - \mathbb{E}[X]) \right] dt \\ &\quad + qdW(t) + \int_Z r(t, \theta)\tilde{\mu}(d\theta, dt), \\ p(T) - \mathbb{E}[p(T)] &= 2G(X(T) - \mathbb{E}[X(T)]), \\ X(0) - \mathbb{E}[X(0)] &= 0. \end{cases} \quad (4.2)$$

In view of the terminal condition of the equations (4.1) and (4.2), now we assume that the state equation $X(\cdot)$ and the adjoint equation $p(\cdot)$ have the following relationship:

$$p(t) = P(t)(X(t) - \mathbb{E}[X(t)]) + \Pi(t)\mathbb{E}[X(t)], \quad (4.3)$$

where $P(\cdot)$ and $\Pi(\cdot)$ taking values in S_+^n are some deterministic differentiable functions such that

$$P(T) = G, \quad \Pi(T) = G + \bar{G}. \quad (4.4)$$

Consequently, we further have the following relationship

$$\mathbb{E}[p(t)] = \Pi(t)\mathbb{E}[X(t)] \quad (4.5)$$

and

$$p(t) - \mathbb{E}[p(t)] = P(t)(X(t) - \mathbb{E}[X(t)]). \quad (4.6)$$

In the following, we begin to formally derive the corresponding Riccati equations which $P(\cdot)$ and $\Pi(\cdot)$ should satisfy. From the relationship (4.6) and (4.2), applying Itô formula to $P(t)(X(t) - \mathbb{E}[X(t)])$ leads to

$$\begin{aligned} & - \left[A^\top (p - \mathbb{E}[p]) + C^\top (q - \mathbb{E}[q]) + \int_Z E^\top(\theta)(r(\theta) - \mathbb{E}[r(\theta)])\nu(d\theta) + 2Q(X - \mathbb{E}[X]) \right] dt \\ & + qdW(t) + \int_Z r(\theta)\tilde{\mu}(d\theta, dt) \\ & = d(p - \mathbb{E}[p]) \\ & = dP(X - \mathbb{E}[X]) \\ & = \left[\dot{P}(X - \mathbb{E}[X]) + P \left(A(X - \mathbb{E}[X]) + B(u - \mathbb{E}[u]) \right) \right] dt \\ & + P \left[C(X - \mathbb{E}[X]) + (C + \bar{C})\mathbb{E}[X] + D(u - \mathbb{E}[u]) + (D + \bar{D})\mathbb{E}[u] \right] dW(t) \\ & + \int_Z P \left[E(\theta)(X - \mathbb{E}[X]) + (E(\theta) + \bar{E}(\theta))\mathbb{E}[X] + F(\theta)(u - \mathbb{E}[u]) + (F(\theta) + \bar{F}(\theta))\mathbb{E}[u] \right] \tilde{\mu}(d\theta, dt). \end{aligned} \quad (4.7)$$

Comparing the diffusion terms of both sides of the above equality, we have

$$q = P \left[C(X - \mathbb{E}[X]) + (C + \bar{C})\mathbb{E}[X] + D(u - \mathbb{E}[u]) + (D + \bar{D})\mathbb{E}[u] \right] \quad (4.8)$$

and

$$r(\theta) = P \left[E(\theta)(X - \mathbb{E}[X]) + (E(\theta) + \bar{E}(\theta))\mathbb{E}[X] + F(\theta)(u - \mathbb{E}[u]) + (F(\theta) + \bar{F}(\theta))\mathbb{E}[u] \right]. \quad (4.9)$$

Then taking expectation on both sides of (4.8) and (4.9), we have the following relationships:

$$\mathbb{E}[q] = P \left[(C + \bar{C})\mathbb{E}[X] + (D + \bar{D})\mathbb{E}[u] \right], \quad (4.10)$$

$$\mathbb{E}[r(\theta)] = P \left[(E(\theta) + \bar{E}(\theta))\mathbb{E}[X] + (F(\theta) + \bar{F}(\theta))\mathbb{E}[u] \right] \quad (4.11)$$

$$q - \mathbb{E}[q] = P \left[C(X - \mathbb{E}[X]) + D(u - \mathbb{E}[u]) \right], \quad (4.12)$$

$$r(\theta) - \mathbb{E}[r(\theta)] = P \left[E(\theta)(X - \mathbb{E}[X]) + F(\theta)(u - \mathbb{E}[u]) \right]. \quad (4.13)$$

In view of (3.1) and (3.8), we get that

$$\begin{aligned} & 2N(u - \mathbb{E}[u]) + B^\top (p - \mathbb{E}[p]) + D^\top (q - \mathbb{E}[q]) \\ & + \int_Z F^\top(\theta)(r(\theta) - \mathbb{E}[r(\theta)])\nu(d\theta) = 0, \quad a.e.a.s. \end{aligned} \quad (4.14)$$

Then putting (4.6), (4.12) and (4.13) into (4.14) yields

$$\begin{aligned}
0 &= N(u - \mathbb{E}[u]) + B^\top P(X - \mathbb{E}[X]) + D^\top P \left[C(X - \mathbb{E}[X]) + D(u - \mathbb{E}[u]) \right] \\
&\quad + \int_Z (F^\top(\theta)P \left[E(\theta)(X - \mathbb{E}[X]) + F(\theta)(u - \mathbb{E}[u]) \right] \nu(d\theta) \\
&= \left[N + D^\top PD + \int_Z F^\top(\theta)PF(\theta)\nu(d\theta) \right] (u - \mathbb{E}[u]) \\
&\quad + \left[B^\top P + D^\top PC + \int_Z F^\top(\theta)PE(\theta)\nu(d\theta) \right] (X - \mathbb{E}[X]) \\
&= \Sigma_0(u - \mathbb{E}[u]) + \left[B^\top P + D^\top PC + \int_Z F^\top(\theta)PE(\theta)\nu(d\theta) \right] (X - \mathbb{E}[X]),
\end{aligned} \tag{4.15}$$

where we denote

$$\Sigma_0 = N + D^\top PD + \int_Z F^\top(\theta)PF(\theta)\nu(d\theta).$$

This implies that

$$u - \mathbb{E}[u] = -\Sigma_0^{-1} \left[B^\top P + D^\top PC + \int_Z (F^\top(\theta)PE(\theta)\nu(d\theta)) \right] \left[X - \mathbb{E}[X] \right]. \tag{4.16}$$

Comparing the drift terms in both sides of (4.7) and combining (4.6), (4.12), (4.13) and (4.16), we get that

$$\begin{aligned}
0 &= (\dot{P} + PA)(X - \mathbb{E}[X]) + PB(u - \mathbb{E}[u]) + A^\top (p - \mathbb{E}[p]) \\
&\quad + C^\top (q - \mathbb{E}[q]) + \int_Z E^\top(\theta)(r(\theta) - \mathbb{E}[r(\theta)])\nu(d\theta) + 2Q(X - \mathbb{E}[X]) \\
&= (\dot{P} + PA)(X - \mathbb{E}[X]) + PB(u - \mathbb{E}[u]) \\
&\quad + A^\top P(X - \mathbb{E}[X]) + C^\top P \left[C(X - \mathbb{E}[X]) + D(u - \mathbb{E}[u]) \right] \\
&\quad + \int_Z E^\top(\theta)P \left[E(\theta)(X - \mathbb{E}[X]) + F(\theta)(u - \mathbb{E}[u]) \right] \nu(d\theta) + 2Q(X - \mathbb{E}[X]) \\
&= \left[(\dot{P} + PA + A^\top P + C^\top PC + \int_Z E^\top(\theta)PE(\theta)\nu(d\theta) + 2Q) \right] (X - \mathbb{E}[X]) \\
&\quad + \left(PB + C^\top PD + \int_Z (E^\top(\theta)PF(\theta)\nu(d\theta)) \right) (u - \mathbb{E}[u]) \\
&= \left[(\dot{P} + PA + A^\top P + C^\top PC + \int_Z E^\top(\theta)PE(\theta)\nu(d\theta) + 2Q) \right] \left[X - \mathbb{E}[X] \right] \\
&\quad - \left[PB + C^\top PD + \int_Z E^\top(\theta)PF(\theta)\nu(d\theta) \right] \Sigma_0^{-1} \\
&\quad \cdot \left[B^\top P + D^\top PC + \int_Z F^\top(\theta)PE(\theta)\nu(d\theta) \right] \left[X - \mathbb{E}[X] \right].
\end{aligned} \tag{4.17}$$

Therefore we should let $P(\cdot)$ be the solution to the following Riccati equation

$$\left\{ \begin{aligned} &(\dot{P} + PA + A^\top P + C^\top PC + \int_Z E^\top(\theta)PE(\theta)\nu(d\theta) + 2Q \\ &\quad - \left[PB + C^\top PD + \int_Z E^\top(\theta)PF(\theta)\nu(d\theta) \right] \Sigma_0^{-1} \\ &\quad \cdot \left[B^\top P(s) + D^\top PC + \int_Z F^\top(\theta)PE(\theta)\nu(d\theta) \right] = 0, \\ &P(T) = G. \end{aligned} \right. \tag{4.18}$$

On the other hand, putting (4.5), (4.10) and (4.11) into (3.8), we get that

$$\begin{aligned}
0 &= 2(N + \bar{N})\mathbb{E}[u] + (B^\top + \bar{B}^\top)\mathbb{E}[p] + (D^\top + \bar{D}^\top)\mathbb{E}[q] + \int_Z (F^\top(\theta) + \bar{F}^\top(\theta))\mathbb{E}[r(\theta)]\nu(d\theta) \\
&= 2(N + \bar{N})\mathbb{E}[u] + (B^\top + \bar{B}^\top)\Pi\mathbb{E}[X] + (D^\top + \bar{D}^\top)P\left[(C + \bar{C})\mathbb{E}[X] + (D + \bar{D})\mathbb{E}[u]\right] \\
&\quad + \int_Z (F^\top(\theta) + \bar{F}^\top(\theta))P\left[(E(\theta) + \bar{E}(\theta))\mathbb{E}[X] + (F(\theta) + \bar{F}(\theta))\mathbb{E}[u]\right] \\
&= \left[2(N + \bar{N}) + (D^\top + \bar{D}^\top)P(D + \bar{D}) + \int_Z (F^\top(\theta) + \bar{F}^\top(\theta))P(F(\theta) + \bar{F}(\theta))\right]\mathbb{E}[u] \\
&\quad + \left[(B^\top + \bar{B}^\top)\Pi + (D^\top + \bar{D}^\top)P(C + \bar{C}) + \int_Z (F^\top(\theta) + \bar{F}^\top(\theta))P(E(\theta) + \bar{E}(\theta))\right]\mathbb{E}[X] \\
&= \Sigma_2\mathbb{E}[u] + \left[(B^\top + \bar{B}^\top)\Pi + (D^\top + \bar{D}^\top)P(C + \bar{C}) + \int_Z (F^\top(\theta) + \bar{F}^\top(\theta))P(E(\theta) + \bar{E}(\theta))\right]\mathbb{E}[X],
\end{aligned} \tag{4.19}$$

where

$$\Sigma_2 := 2(N + \bar{N}) + (D^\top + \bar{D}^\top)P(D + \bar{D}) + \int_Z (F^\top(\theta) + \bar{F}^\top(\theta))P(F(\theta) + \bar{F}(\theta)). \tag{4.20}$$

This implies that

$$\mathbb{E}[u] = -\Sigma_2^{-1}\left[(B^\top + \bar{B}^\top)\Pi + (D^\top + \bar{D}^\top)P(C + \bar{C}) + \int_Z (F^\top(\theta) + \bar{F}^\top(\theta))P(E(\theta) + \bar{E}(\theta))\right]\mathbb{E}[X]. \tag{4.21}$$

Furthermore, from (4.5) and (4.1), we have

$$\begin{aligned}
& - \left[(A^\top + \bar{A}^\top)\mathbb{E}[p] + (C^\top + \bar{C}^\top)\mathbb{E}[q(t)] + \int_Z (E^\top(\theta) + \bar{E}^\top(\theta))\mathbb{E}[r(\theta)]\nu(d\theta) + 2(Q + \bar{Q})\mathbb{E}[X]\right]dt \\
&= d\mathbb{E}[p] \\
&= d\Pi\mathbb{E}[X] \\
&= \left[\dot{\Pi}\mathbb{E}[X] + \Pi(A + \bar{A})\mathbb{E}[X] + \Pi(B + \bar{B})\mathbb{E}[u]\right]dt.
\end{aligned} \tag{4.22}$$

Putting (4.5), (4.10) and (4.11) into the left hand of the above equality and comparing both sides of the above equality, we get

$$\begin{aligned}
0 &= \dot{\Pi}\mathbb{E}[X] + \Pi(A + \bar{A})\mathbb{E}[X] + (B + \bar{B})\mathbb{E}[u] + (A^T + \bar{A}^T)\Pi\mathbb{E}[X] + (C^T + \bar{C}^T)P\left[(C + \bar{C})\mathbb{E}[X] + (D + \bar{D})\mathbb{E}[u]\right] \\
&\quad + \int_Z (E^T(\theta) + \bar{E}^T(\theta))P\left[(E(\theta) + \bar{E}(\theta))\mathbb{E}[X] + (F(\theta) + \bar{F}(\theta))\mathbb{E}[u]\right]\nu(d\theta) + 2(Q + \bar{Q})\mathbb{E}[X(t)] \\
&= \left[\dot{\Pi} + \Pi(t)(A + \bar{A}) + (A^T + \bar{A}^T)\Pi + (C^T + \bar{C}^T)P(C + \bar{C}) + \int_Z (E^T(\theta) + \bar{E}^T(\theta))P(E(\theta) + \bar{E}(\theta)) + 2(Q + \bar{Q})\right]\mathbb{E}[X] \\
&\quad + \left[\Pi(B + \bar{B}) + (C^T + \bar{C}^T)P(D + \bar{D}) + \int_Z (E^T(\theta) + \bar{E}^T(\theta))P(F(\theta) + \bar{F}(\theta))\nu(d\theta)\right]\mathbb{E}[u].
\end{aligned} \tag{4.23}$$

Therefore, by putting (4.21) into (4.23), we conclude that $\Pi(\cdot)$ should be the solutions to the following Riccati equation

$$\left\{ \begin{array}{l} \dot{\Pi} + \Pi(A + \bar{A}) + (A^\top + \bar{A}^\top)\Pi + (C^\top + \bar{C}^\top)P(C + \bar{C}) + \int_Z (E^\top(\theta) + \bar{E}^\top(\theta))P(E(\theta)\nu(d\theta) + \bar{E}(\theta))\nu(d\theta) \\ + 2(Q + \bar{Q}) - \left[\Pi(B + \bar{B}) + (C^\top + \bar{C}^\top)P(D + \bar{D}) + \int_Z (E^\top(\theta) + \bar{E}^\top(\theta))P(F(\theta) + \bar{F}(\theta))\nu(d\theta) \right] \Sigma_2^{-1} \\ \cdot \left[(B^\top + \bar{B}^\top)\Pi + (D^\top + \bar{D}^\top)P(C + \bar{C}) + \int_Z (F^\top(\theta) + \bar{F}^\top(\theta))P(E(\theta) + \bar{E}(\theta))\nu(d\theta) \right] = 0. \\ \Pi = G + \bar{G} \end{array} \right. \quad (4.24)$$

By [28], under Assumptions 1.1 and 1.2, we know that Riccati equations (4.18) and (4.24) has a unique solution, respectively.

4.2 State Feedback Representation

In this section, we establish strictly the link between the stochastic Hamilton system (3.7) and Riccati equation, and show the optimal control has a state feedback representation and the value function are expected to be given in terms of the solution to the Riccati equation. Now we state our main result as follows.

Theorem 4.1. *Let Assumption 1.1 and 1.2 be satisfied. Suppose that $(u(\cdot), X(\cdot), p(\cdot), q(\cdot), r(\cdot, \cdot))$ is the solution to the stochastic Hamilton system (3.7). Let $P(\cdot)$ and $\Pi(\cdot)$ be the solution to the Riccati equations (4.18) and (4.24), respectively. Then the optimal control $u(\cdot)$ has the following state feedback representation:*

$$\begin{aligned} u = & -\Sigma_0^{-1} \left(B^\top P(s) + D^\top PC + \int_Z (F^\top(\theta)PE(\theta)\nu(d\theta)) \right) (X - \mathbb{E}[X]) \\ & - \Sigma_2^{-1} \left[(B^\top + \bar{B}^\top)\Pi + (D^\top + \bar{D}^\top)P(C + \bar{C}) + \int_Z (F^\top(\theta)\nu(d\theta) + \bar{F}^\top(\theta))P(E(\theta)\nu(d\theta) + \bar{E}(\theta)) \right] \mathbb{E}[X] \end{aligned} \quad (4.25)$$

and the following relations hold:

$$p = P(X - \mathbb{E}X) + \Pi \mathbb{E}X, \quad (4.26)$$

$$\begin{aligned} q = & \left[PC - PD\Sigma_0^{-1} \left(B^\top P + D^\top PC + \int_Z (F^\top(\theta)PE(\theta)\nu(d\theta)) \right) \right] (X - \mathbb{E}X) \\ & + \left[P(C + \bar{C}) - P(D + \bar{D})\Sigma_2^{-1} \left((B^\top + \bar{B}^\top)\Pi + (D^\top + \bar{D}^\top)P(C + \bar{C}) \right. \right. \\ & \left. \left. + \int_Z (F^\top(\theta) + \bar{F}^\top(\theta))P(E(\theta) + \bar{E}(\theta)) \right) \right] \mathbb{E}[X], \end{aligned} \quad (4.27)$$

$$\begin{aligned} r(\theta) = & \left[PE(\theta) - PF(\theta)\Sigma_0^{-1} \left(B^\top P + D^\top PC + \int_Z (F^\top(\theta)PE(\theta)\nu(d\theta)) \right) \right] (X - \mathbb{E}X) \\ & + \left[P(E(\theta) + \bar{E}(\theta)) - P(F(\theta) + \bar{F}(\theta))\Sigma_2^{-1} \left((B^\top + \bar{B}^\top)\Pi + (D^\top + \bar{D}^\top)P(C + \bar{C}) \right. \right. \\ & \left. \left. + \int_Z (F^\top(\theta) + \bar{F}^\top(\theta))P(E(\theta) + \bar{E}(\theta)) \right) \right] \mathbb{E}[X]. \end{aligned} \quad (4.28)$$

Moreover,

$$\inf_{v(\cdot) \in \mathcal{A}} J(x, v(\cdot)) = \frac{1}{2} \langle \Pi(0)x, x \rangle. \quad (4.29)$$

Proof. Let $P(\cdot)$ and $\Pi(\cdot)$ be the unique solution to the Riccati equations (4.18) and (4.24), respectively. Consider the following mean-field SDE

$$\begin{cases} dX^*(t) = (A(t)X^*(t) + \bar{A}(t)\mathbb{E}[X^*(t)] + B(t)u^*(t) + \bar{B}(t)\mathbb{E}[u^*(t)])dt \\ \quad + (C(t)X^*(t) + \bar{C}(t)\mathbb{E}[X^*(t)] + D(t)u^*(t) + \bar{D}(t)\mathbb{E}[u^*(t)])dW(t) \\ \quad + \int_Z (E(t, \theta)X^*(t-) + \bar{E}(t, \theta)\mathbb{E}[\hat{X}^*(t-)] + F(t, \theta)u^*(t) + \bar{F}(t, \theta)\mathbb{E}[u^*(t)])\tilde{\mu}(d\theta, dt), \\ x(0) = x \in \mathbb{R}^n, \end{cases} \quad (4.30)$$

where

$$\begin{aligned} u^* = & -\Sigma_0^{-1} \left(B^\top P(s) + D^\top PC + \int_Z (F^\top(\theta)PE(\theta)\nu(d\theta)) \right) (X^* - \mathbb{E}[X^*]) \\ & - \Sigma_2^{-1} \left[(B^\top + \bar{B}^\top)\Pi + (D^\top + \bar{D}^\top)P(C + \bar{C}) + \int_Z (F^\top(\theta)\nu(d\theta) + \bar{F}^\top(\theta))P(E(\theta)\nu(d\theta) + \bar{E}(\theta)) \right] \mathbb{E}[X^*]. \end{aligned} \quad (4.31)$$

The solution to (4.30) is denoted by $X^*(\cdot)$. Define

$$p^* = P(X^* - \mathbb{E}[X^*]) + \Pi\mathbb{E}[X^*], \quad (4.32)$$

$$\begin{aligned} q^* = & \left[PC - PD\Sigma_0^{-1} \left(B^\top P + D^\top PC + \int_Z (F^\top(\theta)PE(\theta)\nu(d\theta)) \right) \right] (X^* - \mathbb{E}[X^*]) \\ & + \left[P(C + \bar{C}) - P(D + \bar{D})\Sigma_2^{-1} \left((B^\top + \bar{B}^\top)\Pi + (D^\top + \bar{D}^\top)P(C + \bar{C}) \right. \right. \\ & \left. \left. + \int_Z (F^\top(\theta) + \bar{F}^\top(\theta))P(E(\theta) + \bar{E}(\theta)) \right) \right] \mathbb{E}[X^*], \end{aligned} \quad (4.33)$$

$$\begin{aligned} r^*(\theta) = & \left[PE(\theta) - PF(\theta)\Sigma_0^{-1} \left(B^\top P + D^\top PC + \int_Z (F^\top(\theta)PE(\theta)\nu(d\theta)) \right) \right] (X^* - \mathbb{E}[X^*]) \\ & + \left[P(E(\theta) + \bar{E}(\theta)) - P(F(\theta) + \bar{F}(\theta))\Sigma_2^{-1} \left((B^\top + \bar{B}^\top)\Pi + (D^\top + \bar{D}^\top)P(C + \bar{C}) \right. \right. \\ & \left. \left. + \int_Z (F^\top(\theta) + \bar{F}^\top(\theta))P(E(\theta) + \bar{E}(\theta)) \right) \right] \mathbb{E}[X^*]. \end{aligned} \quad (4.34)$$

Then following the derivation of riccati equations (4.18) and (4.24) in the previous subsection, by applying Itô formula to $p^* = P(X^* - \mathbb{E}[X^*]) + \Pi\mathbb{E}[X^*]$, we get that $(p^*(\cdot), q^*(\cdot), r^*(\cdot))$ satisfies the following backward stochastic differential equation

$$\begin{cases} dp^*(t) = - \left[A^\top(t)p^*(t) + \bar{A}(t)^\top \mathbb{E}[p^*(t)] + C^\top(t)q^*(t) + \bar{C}(t)^\top \mathbb{E}[q^*(t)] + \int_Z E^\top(t, \theta)r^*(t, \theta)\nu(d\theta) \right. \\ \quad \left. + \int_Z \bar{E}^\top(t, \theta)\mathbb{E}[r^*(t, \theta)]\nu(d\theta) + 2Q(t)X^*(t) + 2\bar{Q}(t)\mathbb{E}[X^*(t)] \right] dt + q^*(t)dW(t) + \int_Z r^*(t, \theta)\tilde{\mu}(d\theta, dt), \\ p^*(T) = 2GX^*(T) + 2\bar{G}\mathbb{E}[X^*(T)], \end{cases} \quad (4.35)$$

which is the adjoint equation associated with $(u^*(\cdot), X^*(\cdot))$. Moreover, it is easy to check that

$$\begin{aligned} 2N(t)u^*(t) + 2\bar{N}(t)\mathbb{E}[u^*(t)] + B^\top(t)p^*(t-) + \bar{B}^\top(t)\mathbb{E}[p^*(t-)] + D^\top(t)q^*(t) + \bar{D}^\top(t)\mathbb{E}[q^*(t)] \\ + \int_Z F^\top(t, \theta)r^*(t, \theta)\nu(d\theta) + \int_Z \bar{F}^\top(t, \theta)\mathbb{E}[r^*(t, \theta)]\nu(d\theta) = 0, \quad a.e.a.s.. \end{aligned} \quad (4.36)$$

Thus, by Theorem 3.1, in terms of (4.36), we know that $u^*(\cdot)$ is the corresponding optimal control and $(u^*(\cdot), X^*(\cdot), p^*(\cdot), q^*(\cdot), r^*(\cdot, \cdot))$ is the solution to the stochastic Hamilton system (3.7). Therefore, from the uniqueness of the solution to the stochastic Hamilton system (3.7), we get that $(u(\cdot), X(\cdot), p(\cdot), q(\cdot), r(\cdot, \cdot)) = (u^*(\cdot), X^*(\cdot), p^*(\cdot), q^*(\cdot), r^*(\cdot, \cdot))$ which implies that (4.25)-(4.28) holds.

Now we begin to prove (4.29). In fact, since $u(\cdot)$ is the optimal control,

$$\begin{aligned} \inf_{v(\cdot) \in \mathcal{A}} J(x, v(\cdot)) &= J(x, u(\cdot)) \\ &= \mathbb{E} \left[\int_0^T \left(\langle Q(t)X(t), X(t) \rangle + \langle \bar{Q}(t)\mathbb{E}[X(t)], \mathbb{E}[X(t)] \rangle + \langle N(t)u(t), u(t) \rangle \right. \right. \\ &\quad \left. \left. + \langle \bar{N}(t)\mathbb{E}[u(t)], \mathbb{E}[u(t)] \rangle \right) dt \right] + \mathbb{E} \langle GX(T), X(T) \rangle + \langle \bar{G}\mathbb{E}[X(T)], \mathbb{E}[X(T)] \rangle. \end{aligned} \quad (4.37)$$

On the other hand, applying Itô formula to $\langle p(t), X(t) \rangle$, we get that

$$\begin{aligned} &2\mathbb{E} \left[\int_0^T \left(\langle Q(t)X(t), X(t) \rangle + \langle \bar{Q}(t)\mathbb{E}[X(t)], \mathbb{E}[X(t)] \rangle \right) dt \right] + 2\mathbb{E} \left[\langle GX(T), X(T) \rangle \right] + 2\langle \bar{G}\mathbb{E}[X(T)], \mathbb{E}[X(T)] \rangle \\ &= \mathbb{E} \left[\int_0^T \left(\langle p(t), B(t)u(t) + \bar{B}(t)\mathbb{E}[u(t)] \rangle + \langle q(t), D(t)u(t) + \bar{D}(t)\mathbb{E}[u(t)] \rangle \right. \right. \\ &\quad \left. \left. + \int_z \langle r(t, \theta), F(t, \theta)u(t) + \bar{F}(t, \theta)\mathbb{E}[u(t)] \rangle \nu(d\theta) \right) dt \right] + \mathbb{E} \langle p(0), x \rangle \\ &= \mathbb{E} \left[\int_0^T \left\langle B^\top(t)p(t-) + \bar{B}^\top(t)\mathbb{E}[p(t-)] + D^\top(t)q(t) + \bar{D}^\top(t)\mathbb{E}[q(t)] \right. \right. \\ &\quad \left. \left. + \int_Z F^\top(t, \theta)r(t, \theta)\nu(d\theta) + \int_Z \bar{F}^\top(t, \theta)\mathbb{E}[r(t, \theta)]\nu(d\theta), u(t) \right\rangle dt \right] + \mathbb{E} \langle p(0), x \rangle. \end{aligned} \quad (4.38)$$

Then putting (4.38) into (4.37) and in terms of (3.1), we get that

$$\begin{aligned} \inf_{v(\cdot) \in \mathcal{A}} J(x, v(\cdot)) &= \frac{1}{2} \langle p(0), x \rangle + \frac{1}{2} \mathbb{E} \left[\int_0^T \left\langle B^\top(t)p(t-) + \bar{B}^\top(t)\mathbb{E}[p(t-)] + D^\top(t)q(t) + \bar{D}^\top(t)\mathbb{E}[q(t)] \right. \right. \\ &\quad \left. \left. + \int_Z F^\top(t, \theta)r(t, \theta)\nu(d\theta) + \int_Z \bar{F}^\top(t, \theta)\mathbb{E}[r(t, \theta)]\nu(d\theta) + 2N(t)u(t) + 2\bar{N}(t)\mathbb{E}[u(t)], u(t) \right\rangle dt \right] \\ &= \frac{1}{2} \langle p(0), x \rangle \end{aligned} \quad (4.39)$$

From (4.26), we have

$$p(0) = \Pi(0)x \quad (4.40)$$

Therefore, putting (4.40) into (4.39) leads to

$$\inf_{u(\cdot) \in \mathcal{A}} J(x, v(\cdot)) = \frac{1}{2} \langle \Pi(0)x, x \rangle. \quad (4.41)$$

The proof is complete. \square

References

- [1] Andersson, D., & Djehiche, B., 2011. A maximum principle for SDEs of mean-field type. *Applied Mathematics and Optimization*, 63, 341-356.

- [2] Bagheri, F., Bagheri, F., & Øksendal, B. (2007). A maximum principle for stochastic control with partial information. *Stochastic Analysis and Applications*, 25(3), 705-717.
- [3] Buckdahn, R., Djehiche, B., & Li, J., 2011. A general stochastic maximum principle for SDEs of mean-field type. *Applied Mathematics and Optimization*, 64, 197-216.
- [4] Djehiche, B., & Tembine, H. (2016). Risk-Sensitive Mean-Field Type Control Under Partial Observation. *In Stochastics of Environmental and Financial Economics (pp. 243-263)*. Springer International Publishing.
- [5] Chala, A. (2014). The relaxed optimal control problem for Mean-Field SDEs systems and application. *Automatica*, 50(3), 924-930.
- [6] Du, H., Huang, J., & Qin, Y. (2013). A stochastic maximum principle for delayed mean-field stochastic differential equations and its applications. *IEEE Transactions on Automatic Control*, 38, 3212-3217.
- [7] Ekeland, I., & Témam, R., 1976. *Convex Analysis and Variational Problems*, North-Holland, Amsterdam.
- [8] Elliott, R., Li, X., & Ni, Y. H, 2013. Discrete time mean-field stochastic linear-quadratic optimal control problems. *Automatica*, 49(11), 3222-3233.
- [9] Hafayed, M. (2013). A mean-field maximum principle for optimal control of forward-backward stochastic differential equations with Poisson jump processes. *International Journal of Dynamics and Control*, 1(4), 300-315
- [10] Hafayed, M., Abbas, S., & Abba, A. (2015). On mean-field partial information maximum principle of optimal control for stochastic systems with Lvy processes. *Journal of Optimization Theory and Applications*, 167(3), 1051-1069.
- [11] Huang, J., Li, X., & Yong, J., 2015. A linear-quadratic optimal control problem for mean-field stochastic differential equations in infinite horizon. *Mathematical Control and Related Fields*.
- [12] Li, J., 2012. Stochastic maximum principle in the mean-field controls. *Automatica*, 48, 366-373.
- [13] Ma, H., & Liu, B. , 2017. Linear Quadratic Optimal Control Problem for Partially Observed Forward Backward Stochastic Differential Equations of Mean-Field Type. *Asian Journal of Control*, 19(1), 1-12.
- [14] Ma, L., & Zhang, W., 2015. Output Feedback H^∞ Control for Discrete time Mean-field Stochastic Systems. *Asian Journal of Control*, 17(6), 2241-2251.
- [15] Ma, L., Zhang, T., & Zhang, W., 2016. H^∞ Control for Continuous Time Mean-Field Stochastic Systems. *Asian Journal of Control*, 18(5), 1630-1640.
- [16] Meng, Q., & Shen, Y., 2015. Optimal control of mean-field jump-diffusion systems with delay: A stochastic maximum principle approach. *Journal of computational and applied mathematics*, 279, 13-30.
- [17] Meyer-Brandis, T., Øksendal, B., Zhou, X.Y., 2012. A mean-field stochastic maximum principle via Malliavin calculus. *Stochastics*, 84, 643-666.
- [18] Ni, Y. H., Li, X., & Zhang, J. F, 2015. Finite-Horizon Indefinite Mean-Field Stochastic Linear-Quadratic Optimal Control. *IFAC-PapersOnLine*, 48(28), 211-216.
- [19] Ni, Y. H., Zhang, J. F., & Li, X, 2015. Indefinite mean-field stochastic linear-quadratic optimal control. *IEEE Transactions on Automatic Control*, 60(7), 1786-1800.
- [20] Zhang, H., & Qi, Q, 2016. A Complete Solution to Optimal Control and Stabilization for Mean-field Systems: Part I, Discrete-time Case. arXiv preprint arXiv:1608.06363.
- [21] Qi, Q., & Zhang, H, 2016. A Complete Solution to Optimal Control and Stabilization for Mean-field Systems: Part II, Continuous-time Case. arXiv preprint arXiv:1608.06475.

- [22] Shen, Y., Meng, Q., & Shi, P., 2014. Maximum principle for mean-field jump-diffusion stochastic delay differential equations and its application to finance. *Automatica*, 50(6), 1565-1579.
- [23] Shen, Y., & Siu, T.K., 2013. The maximum principle for a jump-diffusion mean-field model and its application to the mean-variance problem. *Nonlinear Analysis: Theory, Methods & Applications*, 86, 58-73.
- [24] Wang, G., Wu, Z., & Zhang, C., 2014a. Maximum principles for partially observed mean-field stochastic systems with application to financial engineering. In *Control Conference (CCC), 2014 33rd Chinese (pp. 5357-5362). IEEE*.
- [25] Wang, G., Wu, Z., & Zhang, C., 2016. A partially observed optimal control problem for mean-field type forward-backward stochastic system. In *Control Conference (CCC), 2016 35th Chinese (pp. 1781-1786). TCCT*.
- [26] Wang, G., Xiao, H., & Xing, G., 2015b. A class of optimal control problems for mean-field forward-backward stochastic systems with partial information. *arXiv preprint arXiv:1509.03729*.
- [27] Wang, G., Zhang, C., & Zhang, W., 2014b. Stochastic maximum principle for mean-field type optimal control under partial information. *IEEE Transactions on Automatic Control*, 59(2), 522-528.
- [28] Yong, J., 2013. Linear-quadratic optimal control problems for mean-field stochastic differential equations. *SIAM journal on Control and Optimization*, 51(4), 2809-2838.